[1st Kyu] Section 2: Application Test

1 (Selective)

Consider sums of consecutive positive integers (more than one integer) such as $3+4+5$. Let n be a positive integer.

(1) Prove that n cannot be expressed as a sum of consecutive positive integers if and only if n is a power of 2. (*Proof skill*)

(2) In general, the number of ways of expressing n as the sum of consecutive positive integers is odd. What kind of n, is the number of ways even? Answer with your reasoning.

2 (Selective)

If $f(x)$ is differentiable on the interval $a \le x \le b$ and $f'(x)$ is continuous, then there exists a number c, $a \le c \le b$, such that

$$f(b) - f(a) = (b-a)f'(c).$$

You needn't prove this.

Then, determine the function $f(x)$ defined in $x > 0$ satisfying the conditions below.

$f(x)$ is twice differentiable and $f''(x)$ is continuous. The point c is always the geometric mean of a and b. That is, it satisfies the following equality for arbitrary positive real numbers u and v for $0 < u < v$.

$$f(v) - f(u) = (v-u)f'(\sqrt{uv}).$$

3 (Selective)

There is a tetrahedron ABCD with edge lengths

$$\mathrm{AB} = \mathrm{BC} = \mathrm{CD} = 1, \quad \mathrm{AC} = \mathrm{BD} = \sqrt{2} \text{ and } \mathrm{AD} = \sqrt{3}.$$

Find all the dihedral angles, i.e. angles between two planes, for each edge of the tetrahedron.

4 (Selective)

In the Pre-2nd Practical Mathematics Test, there was the following problem:

> Expand.
> $(x-y)(x+y)(x-2y)$

For this problem, 1067 males had the correct answer out of 1361 males and 694 females had the correct answer out of 851 females. Let the hypothesis be "There is a statistical difference between the percentages of the correct answer for male and female". Conduct the two-tailed hypothesis test at the significance level 0.05. You may assume $z(0.025)=1.96$, where $z(\alpha)$ represents the value of m satisfying $P(X>m)=\alpha$ in random variable X that follows a normal distribution with the mean 0 and the variance 1.

(*Statistical skill*)

5 (Selective)

In a bag there are a total of n black balls and white balls, indistinguishable except for their colors. Let r be the number of black balls and it satisfies $2r<n$. If r balls are chosen at the same time, find the expected value of the number of black balls chosen.

6 (Required)

Let R^n be an n-dimensional vector space and let A be the matrix given by

$$A = \begin{pmatrix} 1 & 3 & -2 & -1 \\ 2 & 5 & -6 & -6 \\ -4 & -13 & 6 & 0 \end{pmatrix}.$$

Define linear map $F:\ R^4 \to R^3$ by $F(x) = Ax$. (*Expression skill*)

(1) Let $\mathrm{Im}\,F = \left\{ F(x) \,\middle|\, x \text{ is an element of } R^4 \right\}$ and let $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be an element of $\mathrm{Im}\,F$. Find the relation among x_1, x_2 and x_3.

(2) Let $\mathrm{Ker}\,F = \left\{ x \,\middle|\, x \text{ is an element of } R^4 \text{ and } F(x) = O \right\}$, where O represents the three-dimensional null vector. Let $\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$ be an element of $\mathrm{Ker}\,F$. Find the general form of $\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$ when $y_1 + y_2 + y_3 + y_4 = 0$.

7 (Required)

A fixed point lie on the surface of a sphere of radius r. Find the area of the spherical cap within the distance being less than or equal to a from the fixed point, where a is a constant satisfying $0 < a < \pi r$. Here, the distance between 2 points on the spherical surface is regarded as the arc of a circle of radius r through the 2 points.