1

(1) The sum of k consecutive positive integers starting from a, $S(a,\ k)$, can be expressed as

$$S(a,\ k)=ak+\frac{k(k-1)}{2},$$

where $S(a,\ 1)=a$ for convenience.

If $n=S(a,\ k)$, then $2n=k(k+2a-1)$. So the value on the right side is even. Remark that k and $k+2a-1$ have opposite parity. Let

$$2n=2^e\times m \quad (m \text{ is odd}).$$

Express m as the product of positive divisors such that $2n=r\times s$ (r is even and s is odd).

For r and s, Let k be the smaller one and let $k+2a-1$ be the larger one. Then the formula expresses a sum of consecutive integers except when $k=1$.

If $n=2^\ell$, then $2n=2^{\ell+1}\times 1$ and $k=1$ in the above formula, i.e. it is not the sum of consecutive integers. If n is not a power of 2, it has odd divisor greater than 2, and when it is expressed as $2n=r\times s$, we see $r\geq 2$ and $s\geq 3$. Hence when k and $k+2a-1$ are determined as we saw, we have $k\geq 2$ and we can express it as a sum of consecutive positive integers. Therefore n cannot be expressed as a sum of consecutive positive integers if and only if n is a power of 2.

(2) If n is not a power of 2, we factor in prime numbers as follows:

$$m=p_1^{e_1}\times\cdots\times p_n^{e_n}$$

where $p_1,\ \cdots,\ p_n$ are odd primes. Here, the formula $2n=r\times s$ and the formulae

$$r=2^{e+1}p,\quad s=q$$

given by the product of $m=pq$ and divisors have one-to-one correspondence. However, if $q=1$, then $k=1$, so it must be rejected. If $q\geq 3$, it gives a sum of consecutive integers. Therefore the number of ways of expressing sums of consecutive integers is the number of divisors of m other than 1, i.e.

$$(e_1+1)\cdots\cdots(e_\ell+1)-1.$$

This value is even if and only if $e_1,\ \cdots$ and e_ℓ are all even. Then m is a square number. Let m be ℓ^2, we have

$$n=2^e\times\ell^2 \quad (\ell \text{ is odd other than 1})$$

(Answer) $n=2^e\times\ell^2$ (ℓ is odd other than 1)

2

Let u be fixed and differentiate both sides of $f(v)-f(u)=(v-u)f'(\sqrt{vu})$ with respect to v.

$$f'(v)=f'(\sqrt{vu})+(v-u)\frac{\sqrt{u}}{2\sqrt{v}}f''(\sqrt{uv})$$

Here, let A be a positive constant small enough and let

$u=\dfrac{A^2}{v}$, then we have $0<u<v$ when $v>A$ and

$$f'(v)=f'(A)+\left(v-\frac{A^2}{v}\right)\frac{A}{2v}f''(A)$$

$$=f'(A)+\frac{Af''(A)}{2}\left(1-\frac{A^2}{v^2}\right)$$

Here, let

$$a=\frac{A^3f''(A)}{2} \text{ and } b=f'(A)+\frac{Af''(A)}{2}.$$

We have

$$f'(v)=-\frac{a}{v^2}+b.$$

That is,

$$f(v)=\frac{a}{v}+bv+c \quad (a,\ b,\ c \text{ are constants})$$

must be held to satisfy the conditions. Conversely the function

$$f(x)=\frac{a}{x}+bx+c,$$

satisfies the given condition.

(Answer) $f(x)=\dfrac{a}{x}+bx+c$

($a,\ b,\ c$ are constants)

3

Because $AB^2+BC^2=CA^2$, $BC^2+CD^2=DB^2$ and $AB=BC=CD$, △ABC and △BCD are right angled isosceles triangles. △ABD and △ACD are right angled triangles whose ratio of sides is $1:\sqrt{2}:\sqrt{3}$. That is $\angle ABC=\angle ABD=\angle ACD=\angle DCB=90^\circ$.

Let us consider the tetrahedron in coordinate space with the origin D. We may choose DC, CB and BA be the positive x-axis, y-axis and z-axis, respectively, and we have C$(1, 0, 0)$, B$(1, 1, 0)$ and $(1, 1, 1)$.

The surface equations of △ABC, △ACD, △ABD and △BCD can be expressed as

$x=1$, $y=z$, $x=y$, $z=0$,

respectively. Their normal vectors are

$\boldsymbol{p}=(1, 0, 0)$, $\boldsymbol{q}=(0, -1, 1)$,

$\boldsymbol{r}=(1, -1, 0)$ and $\boldsymbol{s}=(0, 0, 1)$,

respectively. Therefore, the dihedral angles are shown below with their normal vectors.

Edge	Normal vector	Dot product	Cosine of dihedral angles	Dihedral angle
AB	$\boldsymbol{p}$ and $\boldsymbol{r}$	1	$\frac{1}{\sqrt{2}}$	45°
BC	$\boldsymbol{p}$ and $\boldsymbol{s}$	0	0	90°
CD	$\boldsymbol{q}$ and $\boldsymbol{s}$	1	$\frac{1}{\sqrt{2}}$	45°
AC	$\boldsymbol{p}$ and $\boldsymbol{q}$	0	0	90°
BD	$\boldsymbol{r}$ and $\boldsymbol{s}$	0	0	90°
AD	$\boldsymbol{q}$ and $\boldsymbol{r}$	1	$\frac{1}{2}$	60°

(Answer) edge AB⋯45°, edge BC⋯90°,
edge CD⋯45°, edge AC⋯90°,
edge BD⋯90°, edge AD⋯60°.

4

Let H_0 and H_1 be the following:

- Null hypothesis H_0 : There is no statistical difference between the percentages of the correct answer for male and female.
- Alternate hypothesis H_1 : There is a statistical difference between the percentages of the correct answer for male and female.

We conduct the two-tailed hypothesis test at the significance level 0.05.

Since 1761 out of 2212 people got the correct answer, the statistic Z_0 is

$$Z_0=\frac{\left|\frac{1067}{1361}-\frac{694}{851}\right|}{\sqrt{\frac{1761}{2212}\cdot\frac{451}{2212}\cdot\left(\frac{1}{1361}+\frac{1}{851}\right)}}$$

$$=1.7907\cdots.$$

Because $Z_0<1.96=z(0.025)$, we cannot reject H_0.

Hence, we should not conclude that there is a statistical difference between the percentages of the correct answer for male and female.

(Answer) We should not conclude that there is a statistical difference between the percentages of the correct answer for male and female.

5

Let us consider the probability P_k that there are k black balls when choosing r balls, where $0 \leq k \leq r$.
The total number of combination when choosing r balls at the same time from n balls is

$${}_nC_r \left(= \frac{n\,!}{r\,! \cdot (n-r)!}\right).$$

If exactly k black balls are chosen, the total number of combinations of black balls and white balls are ${}_rC_k$ and ${}_{n-r}C_{r-k}$, respectively. So we get

$$P_k = \frac{{}_rC_k \cdot {}_{n-r}C_{r-k}}{{}_nC_r}.$$

The required expected value is

$$E = \sum_{k=0}^{r} P_k \cdot k.$$

Here, since

$$k \cdot {}_rC_k = \frac{r\,! \cdot k}{(r-k)! \cdot k!} = \frac{r \cdot (r-1)!}{(r-k)! \cdot (k-1)!}$$

$$= r \cdot {}_{r-1}C_{k-1},$$

we have

$$E = \frac{r}{{}_nC_r} \sum_{k=1}^{r} ({}_{n-r}C_{r-k} \cdot {}_{r-1}C_{k-1}).$$

Now, the sum $\sum_{k=1}^{r} ({}_{n-r}C_{r-k} \cdot {}_{r-1}C_{k-1})$ equals the coefficient of x^{r-1}-term in binomial expression,

$$(1+x)^{n-r}\,(1+x)^{r-1}$$

$$= \left(\sum_{p=0}^{n-r} {}_{n-r}C_{r-p}\, x^{r-p}\right)\left(\sum_{q=1}^{r} {}_{r-1}C_{q-1}\, x^{q-1}\right).$$

That is, it equals the coefficient of x^{r-1}-term in $(1+x)^{n-1}$. Hence

$$\sum_{k=1}^{r} ({}_{n-r}C_{r-k} \cdot {}_{r-1}C_{k-1}) = {}_{n-1}C_{r-1}.$$

Therefore we get

$$E = \frac{r}{{}_nC_r}\ {}_{n-1}C_{r-1}$$

$$= \frac{r \cdot (n-1)! \cdot r! \cdot (n-r)!}{n! \cdot (r-1)! \cdot (n-r)!}$$

$$= \frac{r \cdot (n-1)! \cdot r \cdot (r-1)!}{n \cdot (n-1)! \cdot (r-1)!}$$

$$= \frac{r^2}{n}$$

(Answer) $\dfrac{r^2}{n}$

6

(1) Using elementary row transformation for A,

$$A \to \begin{pmatrix} 1 & 3 & -2 & -1 \\ 0 & -1 & -2 & -4 \\ 0 & -1 & -2 & -4 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & -8 & -13 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then the rank of A, i.e. the dimension of $\mathrm{Im}F$ is 2. We may choose the basis of $\mathrm{Im}F$ as the first two columns of A, say

$$\left\langle \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \\ -13 \end{pmatrix} \right\rangle,$$

which means the equation

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} + \beta \begin{pmatrix} 3 \\ 5 \\ -13 \end{pmatrix},$$

with real constants α and β. The required linear relation among x_i's

$$\lambda x_1 + \mu x_2 + \nu x_3 = 0$$

is given by the relation

$$\lambda(\alpha+3\beta) + \mu(2\alpha+5\beta) + \nu(-4\alpha-13\beta) = 0$$

with respect to λ, μ and ν. This gives

$$\lambda + 2\mu - 4\nu = 0 \text{ and } 3\lambda + 5\mu - 13\nu = 0.$$

Eliminating λ, we see $\mu + \nu = 0$, and then $\lambda = 6\nu$ and $\mu = -\nu$. Since the relation requires only the ratio $\lambda : \mu : \nu$, we may take it to be

$$6x_1 - x_2 + x_3 = 0.$$

(Answer) $6x_1 - x_2 + x_3 = 0$

(2) From (1), we have the following relations for the elements of $\mathrm{Ker}F$

$$\begin{cases} y_1 \quad -8y_3 - 13y_4 = 0 \cdots ① \\ \quad y_2 + 2y_3 + \ 4y_4 = 0 \cdots ② \end{cases}$$

Here, note that

$$y_1 + y_2 + y_3 + y_4 = 0 \cdots ③$$

From ③−①−②, we get

$$7y_3 + 10y_4 = 0.$$

And $y_4 = -\dfrac{7}{10}y_3$. For ① and ②, we have

$$y_1 = 8y_3 + 13y_4 = -\frac{11}{10}y_3 \text{ and}$$

$$y_2 = -2y_3 - 4y_4 = \frac{8}{10}y_3.$$

Hence by letting $y_3 = 10C$, where C is a real constant, we can express

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = C \begin{pmatrix} -11 \\ 8 \\ 10 \\ -7 \end{pmatrix}.$$

(Answer) $\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = C \begin{pmatrix} -11 \\ 8 \\ 10 \\ -7 \end{pmatrix}$

(C is an arbitrary real constant)

7

We have $a = \alpha r$, where r is a radius, a is the length of an arc and α is the central angle in radian.

Let us consider the spherical coordinates (ρ, θ, ϕ), where the center of the sphere is the origin. The spherical cap within the distance being less than or equal to a from the north pole can be expressed as

$$\rho = r, \quad 0 \le \theta \le \alpha, \quad 0 \le \phi \le 2\pi.$$

Hence the required area is

$$\int_{\phi=0}^{2\pi} \int_{\theta=0}^{\alpha} r^2 \sin\theta \, d\theta d\phi$$

$$= 2\pi r^2 \int_{\theta=0}^{\alpha} \sin\theta \, d\theta$$

$$= 2\pi r^2 (1 - \cos\alpha)$$

$$= 2\pi r^2 \left(1 - \cos\frac{a}{r}\right)$$

(Answer) $2\pi r^2 \left(1 - \cos\dfrac{a}{r}\right)$