1

$$x^2+2xy-y^2+2mx+(6-m)y+4=0 \quad (*)$$

$$x^2+2(y+m)x-y^2+(6-m)y+4=0.$$

Solving this equation for x gives

$$x=-(y+m)\pm\sqrt{(y+m)^2-\{-y^2+(6-m)y+4\}}$$

$$=-y-m\pm\sqrt{2y^2+3(m-2)y+m^2-4}. \quad (**)$$

If (*) represents two lines in the xy-plane, the value inside the radical in (**), $2y^2+3(m-2)y+m^2-4$, must be expressed in the form $(ay+b)^2$, where a and b are real numbers.

It follows that the discriminant D of the quadratic equation $2y^2+3(m-2)y+m^2-4=0$ must be 0.

Since

$$D=9(m-2)^2-8(m^2-4)$$

$$=m^2-36m+68$$

$$=(m-2)(m-34),$$

we have $m=2,\ 34$.

Conversely, for $m=2$, (*) becomes

$$x^2+2xy-y^2+4x+4y+4=0$$

$$\{x+(1+\sqrt{2})y+2\}\{x+(1-\sqrt{2})y+2\}=0. \quad (***)$$

Hence, (***) represents the lines

$$x+(1+\sqrt{2})y+2=0 \text{ and } x+(1-\sqrt{2})y+2=0.$$

For $m=34$, (*) becomes

$$x^2+2xy-y^2+68x-28y+4=0$$

$$\{x+(1+\sqrt{2})y+34+24\sqrt{2}\}\{x+(1-\sqrt{2})y+34-24\sqrt{2}\}=0. \quad (****)$$

Hence, (****) represents the lines

$$x+(1+\sqrt{2})y+34+24\sqrt{2}=0 \text{ and } x+(1-\sqrt{2})y+34-24\sqrt{2}=0.$$

Therefore, the required value of m is 2 or 34.

(Answer) $m=2,\ 34$

2

(1)

$$\frac{a_n a_{n+1} - 2a_n a_{n+2} + a_{n+1}a_{n+2}}{a_n a_{n+1} a_{n+2}} = n$$

$$\frac{1}{a_{n+2}} - \frac{2}{a_{n+1}} + \frac{1}{a_n} = n$$

$$\frac{1}{a_{n+2}} - \frac{1}{a_{n+1}} - \left(\frac{1}{a_{n+1}} - \frac{1}{a_n}\right) = n.$$

Since $b_n = \frac{1}{a_{n+1}} - \frac{1}{a_n}$, we have

$$b_{n+1} - b_n = n.$$

(Answer) $b_{n+1} - b_n = n$

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(2) We have $b_1 = \frac{1}{a_2} - \frac{1}{a_1} = 0$. Since $b_{n+1} - b_n = n$, for $n \geq 2$, we have

$$b_n = b_1 + \sum_{k=1}^{n-1} k$$

$$= \frac{1}{2}(n-1)n$$

$$= \frac{1}{2}(n^2 - n). \quad (*)$$

For $n = 1$, the value of (*) is $\frac{1}{2}(1^2 - 1) = 0 = b_1$. Hence, (*) holds for $n = 1$. Hence, we have

$$\frac{1}{a_{n+1}} - \frac{1}{a_n} = \frac{1}{2}(n^2 - n).$$

For $n \geq 2$, we have

$$\frac{1}{a_n} = \frac{1}{a_1} + \sum_{k=1}^{n-1} \frac{1}{2}(k^2 - k)$$

$$= 1 + \frac{1}{2}\left\{\frac{1}{6}(n-1)n(2n-1) - \frac{1}{2}(n-1)n\right\}$$

$$= \frac{n^3 - 3n^2 + 2n + 6}{6}.$$

Since $n^3 - 3n^2 + 2n + 6 = n(n-1)(n-2) + 6 > 0$, we obtain

$$a_n = \frac{6}{n^3 - 3n^2 + 2n + 6}. \quad (**)$$

For $n = 1$, the value of (**) is $\frac{6}{1^3 - 3 \cdot 1^2 + 2 \cdot 1 + 6} = 1 = a_1$. Hence, (**) holds for $n = 1$.

Therefore, for all positive integers n, we have $a_n = \frac{6}{n^3 - 3n^2 + 2n + 6}$.

(Answer) $a_n = \frac{6}{n^3 - 3n^2 + 2n + 6}$

3

(1) $$f(g(x)) = \frac{g(x)+1}{2g(x)+3} = \frac{\dfrac{ax+b}{cx+d}+1}{2\cdot\dfrac{ax+b}{cx+d}+3} = \frac{(a+c)x+b+d}{(2a+3c)x+2b+3d}.$$

If $f(g(x)) = x+1$ is an identity in x, then

$$(a+c)x+b+d = (2a+3c)x^2+(2a+2b+3c+3d)x+2b+3d$$

is also an identity in x.
Equating the coefficients, we have

$$0 = 2a+3c,\quad a+c = 2a+2b+3c+3d,\quad b+d = 2b+3d.$$

Hence, we have $a=-3d$, $b=-2d$ and $c=2d$ ($d \neq 0$). Therefore, we obtain

$$g(x) = \frac{-3dx-2d}{2dx+d} = -\frac{3x+2}{2x+1}.$$

(Answer) $g(x) = -\dfrac{3x+2}{2x+1}$

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(2) $$g(f(x)) = \frac{af(x)+b}{cf(x)+d} = \frac{a\cdot\dfrac{x+1}{2x+3}+b}{c\cdot\dfrac{x+1}{2x+3}+d} = \frac{(a+2b)x+a+3b}{(c+2d)x+c+3d}.$$

If $g(f(x)) = x+1$ is an identity in x, then

$$(a+2b)x+a+3b = (c+2d)x^2+(2c+5d)x+c+3d$$

is also an identity in x.
Equating the coefficients, we have

$$0 = c+2d,\quad a+2b = 2c+5d,\quad a+3b = c+3d.$$

Hence, we have $a=d$, $b=0$ and $c=-2d$ ($d \neq 0$). Therefore, we obtain

$$g(x) = \frac{dx}{-2dx+d} = \frac{x}{1-2x}.$$

(Answer) $g(x) = \dfrac{x}{1-2x}$

4

(1) (Answer) $3 < x \leq 5$

---

(2) Since $\dfrac{x-3}{2} = \dfrac{2t}{t^2+1}$ and $y = \dfrac{1-t^2}{t^2+1}$, we have

$$\left(\frac{x-3}{2}\right)^2 + y^2 = \left(\frac{2t}{t^2+1}\right)^2 + \left(\frac{1-t^2}{t^2+1}\right)^2 = 1.$$

It follows that

$$\frac{(x-3)^2}{4} + y^2 = 1.$$

From the answer in (1), the locus of point P is a portion of the ellipse $\dfrac{(x-3)^2}{4} + y^2 = 1$ for $3 < x \leq 5$.

(Answer) Portion of the ellipse $\dfrac{(x-3)^2}{4} + y^2 = 1$ for $3 < x \leq 5$

5

(1) The union of $3k$ ($k > 0$), $3k+1$ ($k \geq 0$) and $3k+2$ ($k \geq 0$) contains all positive integers. Hence, there exists n such that $f(n) = N$ for an arbitrary positive integer N.

We prove that exactly one n exists for an arbitrary positive integer N. The remainders when dividing each of $3k$, $3k+1$ and $3k+2$ by 3 are distinct, and if $3k = 3k'$, $3k+1 = 3k'+1$ or $3k+2 = 3k'+2$, then $k = k'$. Hence, we see that if $f(n) = f(n')$, then $n = n'$. It follows that if there is n such that $f(n) = N$, there must be exactly one n.

Therefore, there is exactly one n such that $f(n) = N$ for an arbitrary positive integer N.

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(2) For one-digit positive integers, we have

$$1 \to 1 \to 1 \to \cdots,$$
$$2 \to 3 \to 2 \to 3 \to \cdots,$$
$$4 \to 6 \to 9 \to 7 \to 5 \to 4 \to 6 \to 9 \to 7 \to 5 \to \cdots.$$

Therefore, there is no number that leads to 8 or comes from 8.

(Answer) 0

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(3) (Answer) $m = 73$

6

(1) Since $\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{BC}$, we have

$$\overrightarrow{OD} = \vec{a} - \vec{b} + \vec{c}.$$

For square pyramid O-ABCD, drawing perpendicular line OI from point O to base ABCD, we have

$$\overrightarrow{ON} = \frac{|\overrightarrow{OI}| + |\overrightarrow{AE}|}{|\overrightarrow{OI}|}\overrightarrow{OI}.$$

Since $\overrightarrow{OI} = \dfrac{\vec{a}+\vec{c}}{2}$, $|\overrightarrow{OI}| = \sqrt{2}$ and $|\overrightarrow{AE}| = 2$, we obtain

$$\overrightarrow{ON} = \frac{\sqrt{2}+2}{\sqrt{2}}\left(\frac{\vec{a}+\vec{c}}{2}\right)$$

$$= \frac{1+\sqrt{2}}{2}(\vec{a}+\vec{c}).$$

(Answer) $\overrightarrow{OD} = \vec{a} - \vec{b} + \vec{c}$, $\overrightarrow{ON} = \dfrac{1+\sqrt{2}}{2}(\vec{a}+\vec{c})$

---

(2) Since point P lies on line MN, there is a real number t such that

$$\overrightarrow{OP} = (1-t)\overrightarrow{OM} + t\overrightarrow{ON}$$

$$= \frac{1-t}{2}\overrightarrow{OD} + t\overrightarrow{ON}$$

$$= \frac{1-t}{2}(\vec{a} - \vec{b} + \vec{c}) + \frac{1+\sqrt{2}}{2}t(\vec{a} + \vec{c})$$

$$= \frac{1+\sqrt{2}t}{2}\vec{a} - \frac{1-t}{2}\vec{b} + \frac{1+\sqrt{2}t}{2}\vec{c}.$$

Also, point P lies on the plane that passes through three points A, B and C, we have

$$\frac{1+\sqrt{2}t}{2} - \frac{1-t}{2} + \frac{1+\sqrt{2}t}{2} = 1$$

$$(2\sqrt{2}+1)t = 1$$

$$t = \frac{2\sqrt{2}-1}{7}.$$

Therefore, we obtain

$$\overrightarrow{OP} = \frac{11-\sqrt{2}}{14}\vec{a} - \frac{4-\sqrt{2}}{7}\vec{b} + \frac{11-\sqrt{2}}{14}\vec{c}.$$

(Answer) $\overrightarrow{OP} = \dfrac{11-\sqrt{2}}{14}\vec{a} - \dfrac{4-\sqrt{2}}{7}\vec{b} + \dfrac{11-\sqrt{2}}{14}\vec{c}$

7

The coordinates of the points of intersection of the two curves satisfy the following system of equations.

$$\begin{cases} x^2 + y^2 = 4 & ① \\ \dfrac{x^2}{2} + \dfrac{y^2}{6} = 1. & ② \end{cases}$$

From ①−②×6, we have

$$-2x^2 = -2$$
$$x^2 = 1.$$

Hence, we have $y^2 = 3$.

Therefore, the coordinates of the points of intersection of the two curves are

$$(1,\ \sqrt{3}),\quad (1,\ -\sqrt{3}),\quad (-1,\ \sqrt{3}),\quad (-1,\ -\sqrt{3}).$$

① and ② can be expressed as

$$x^2 = 4 - y^2 \quad ③$$
$$x^2 = 2 - \frac{y^2}{3}. \quad ④$$

Using symmetry of the figure and ③ and ④, the required volume V is

$$V = 2\left\{\pi\int_0^{\sqrt{3}}\left(2 - \frac{y^2}{3}\right)dy + \pi\int_{\sqrt{3}}^{2}(4 - y^2)dy\right\}$$
$$= 2\pi\left(\left[2y - \frac{y^3}{9}\right]_0^{\sqrt{3}} + \left[4y - \frac{y^3}{3}\right]_{\sqrt{3}}^{2}\right)$$
$$= 2\pi\left\{2\sqrt{3} - \frac{\sqrt{3}}{3} + \left(8 - \frac{8}{3}\right) - (4\sqrt{3} - \sqrt{3})\right\}$$
$$= \frac{32 - 8\sqrt{3}}{3}\pi.$$

(Answer) $V = \dfrac{32 - 8\sqrt{3}}{3}\pi$