1st Kyu

Section 2 : Application Test

数学検定

PROFICIENCY TEST
IN
PRACTICAL MATHEMATICS

Test Time : 120 minutes

Test Instructions

1. Make sure that you have the correct level (Kyu) test.
2. Do not open the booklet until you are told to do so.
3. Write your examinee number and name on this page.
4. Write your name, examinee number and other necessary information on the answer sheets.
5. Write your answers on the answer sheets (they are numbered 1 through 4). Write the steps leading to your answer. However if there are specific instructions for a problem, follow the instructions.
6. Problems 1 to 5 are selective problems. Choose two problems from the selective problems and fill in ⬯ to indicate which problems you chose. Then write your answers. Note that all of your answers will not be marked if you answered more than two problems from the selective problems. Problems 6 and 7 are required problems.
7. You may use a calculator.
8. Turn off your cell phone and do not use it during the test.
9. Ask an examination supervisor if your problem sheets have inconsistent page numbering or missing pages.
10. It is prohibited to disclose the problems to the general public, such as on the Internet, without permission.

Examinee Number	–	Name	

※Your personal information will be handled appropriately according to the "Handling of Personal Information" agreement that was approved at the time of registration.

公益財団法人
日本数学検定協会
The Mathematics Certification Institute of Japan

[1st Kyu] Section 2: Application Test

1 (Selective)

For a positive integer n, let $\varphi(n)$ be the number of positive integers less than or equal to n that are relatively prime to n.

(1) Find the value of $\varphi(2021)$. Write only your answer.

(2) Among positive integers x that satisfy the congruence

$$x^{2273} \equiv 5 \pmod{2021},$$

there is only one value of x that is less than or equal to 2021 (you don't need to prove this). Find the integer. You may use the following facts without proving them.

- If two positive numbers a and m are relatively prime, then

  $$a^{\varphi(m)} \equiv 1 \pmod{m}.$$

- The following congruences hold:

  $5^5 \equiv 1104 \pmod{2021}$,
  $5^6 \equiv 1478 \pmod{2021}$,
  $5^7 \equiv 1327 \pmod{2021}$,
  $5^8 \equiv 572 \pmod{2021}$,
  $5^9 \equiv 839 \pmod{2021}$,
  $5^{10} \equiv 153 \pmod{2021}$.

2 (Selective)

The function defined by

$$\Gamma(t) = \int_0^{\infty} x^{t-1} e^{-x}\, dx$$

is called the gamma function, where t is a positive real number and e is the base of the natural logarithm. You many use the following facts without proving.

$$\lim_{x\to\infty} x^a e^{-x} = 0 \quad (a \text{ is a real number}) \quad \text{and} \quad \int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}.$$

(1) Prove the following ① and ②. (*Proof skill*)

① $\Gamma(t+1) = t\Gamma(t)$

② $\Gamma(n+1) = n!$ for a positive integer n

(2) Calculate the following improper integral.

$$\int_0^1 t^3 \left(\log_e \frac{1}{t}\right)^{\frac{5}{2}} dt$$

3 (Selective)

In $\triangle$ABC, let BC $= a$, CA $= b$, AB $= c$, $\angle$CAB $= A$, $\angle$ABC $= B$ and $\angle$BCA $= C$.

If $a+b+c=3$, find the minimum value of

$$T = \frac{a^2}{3+2\sqrt{3}\sin A} + \frac{b^2}{3+2\sqrt{3}\sin B} + \frac{c^2}{3+2\sqrt{3}\sin C}.$$

4 (Selective)

Three people, A, B and C bowled. Person A played 100 frames, person B played 120 frames and person C played 80 frames. The table shows the number of strikes (all ten pins are knocked down on the first roll), the number of spares (the last of the ten pins is/are knocked down on the second roll of a frame) and the number of the other results for each person.

	A	B	C	Total
Strike	27	45	18	90
Spare	18	18	24	60
Other	55	57	38	150
Total	100	120	80	300

Can it be concluded that there is no difference in the proportions of the numbers of strikes, spares and the others for the three people? Conduct the hypothesis test at the significance level 0.05 under

the null hypothesis H_0: there is no difference among the three people,

the alternative hypothesis H_1: there is a difference among the three people.

Use the values in the χ^2-distribution table given. (*Statistical skill*)

5 (Selective)

Let **N** be a set of positive integers. A mapping, denoted by $n = f(m)$, from **N** to **N** is defined as follows:

For an integer k,
(I) If m is even, that is $m = 2k$, $n = f(m) = 3k$.
(II) If m is odd and is in the form $m = 4k+1$, $n = f(m) = 3k+1$.
(III) If m is odd and is in the form $m = 4k+3$, $n = f(m) = 3k+2$.

Answer the following. (*Organizing skill*)

(1) Verify that the mapping f is bijective from **N** to **N** and express the inverse mapping, denoted by $m = f^{-1}(n)$, of $n = f(m)$ in the form shown above.

(2) Letting m_0 be an initial value, form the sequence

$$m_0,\ m_1,\ m_2, \dots$$

by

$$m_{j+1} = f(m_j).$$

Sometimes, depending on the value of m_0, we have

$m_\ell = m_0$, where ℓ is a positive integer.

Here, we call the minimum positive integer ℓ such that $m_\ell = m_0$ the period for the initial value m_0.
For example, if $m_0 = 1$, we have $m_1 = 1$, $m_2 = 1$, …, then the period for the initial value 1 is 1.
Among initial values $m_0 (\neq 1)$ with a period less than or equal to 30, give three numbers of different periods and find their periods for the initial values. Write only your answer.

6 (Required)

Let n be a positive integer and let I be the 3×3 unit matrix. For the 3×3 square matrix

$$A = \begin{pmatrix} -3 & -3 & -5 \\ 3 & 3 & 7 \\ 1 & 1 & 1 \end{pmatrix},$$

A^n is expressed as

$$A^n = p_n A^2 + q_n A + r_n I ,$$

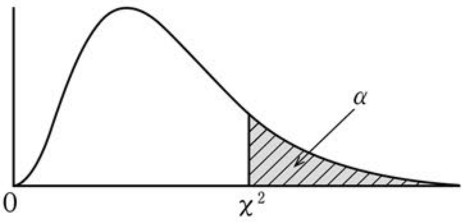
where p_n, q_n and r_n are real numbers. Find p_n, q_n and r_n. (*Expression skill*)

7 (Required)

Find the solution of the following differential equation under the initial conditions $y(0) = -4$ and $y'(0) = 12$, and express it in the form $y = f(x)$.

$$y'' + 6y' + 5y = 26\cos x + 25x$$

The χ^2-Distribution Table (values of α with n degrees of freedom)



The χ^2-Distribution Table

n \ α	0.99	0.975	0.95	0.05	0.025	0.01
1	0.000157	0.000982	0.003932	3.8415	5.0239	6.6349
2	0.020101	0.050636	0.10259	5.9915	7.3778	9.2103
3	0.11483	0.21580	0.35185	7.8147	9.3484	11.345
4	0.29711	0.48442	0.71072	9.4877	11.143	13.277
5	0.55430	0.83121	1.1455	11.070	12.833	15.086
6	0.87209	1.2373	1.6354	12.592	14.449	16.812
7	1.2390	1.6899	2.1673	14.067	16.013	18.475
8	1.6465	2.1797	2.7326	15.507	17.535	20.090
9	2.0879	2.7004	3.3251	16.919	19.023	21.666
10	2.5582	3.2470	3.9403	18.307	20.483	23.209
11	3.0535	3.8157	4.5748	19.675	21.920	24.725
12	3.5706	4.4038	5.2260	21.026	23.337	26.217
13	4.1069	5.0088	5.8919	22.362	24.736	27.688
14	4.6604	5.6287	6.5706	23.685	26.119	29.141
15	5.2293	6.2621	7.2609	24.996	27.488	30.578
16	5.8122	6.9077	7.9616	26.296	28.845	32.000
17	6.4078	7.5642	8.6718	27.587	30.191	33.409
18	7.0149	8.2307	9.3905	28.869	31.526	34.805
19	7.6327	8.9065	10.117	30.144	32.852	36.191
20	8.2604	9.5908	10.851	31.410	34.170	37.566
21	8.8972	10.283	11.591	32.671	35.479	38.932
22	9.5425	10.982	12.338	33.924	36.781	40.289
23	10.196	11.689	13.091	35.172	38.076	41.638
24	10.856	12.401	13.848	36.415	39.364	42.980
25	11.524	13.120	14.611	37.652	40.646	44.314
26	12.198	13.844	15.379	38.885	41.923	45.642
27	12.879	14.573	16.151	40.113	43.195	46.963
28	13.565	15.308	16.928	41.337	44.461	48.278
29	14.256	16.047	17.708	42.557	45.722	49.588
30	14.953	16.791	18.493	43.773	46.979	50.892