1

(1) (Answer) 1932

(2) For the equation

$$2273a - 1932b = 1,$$

we find one pair of positive integers a and b that satisfy the equation as follows:
Since

$$2273 = 1932 \cdot 1 + 341$$
$$1932 = 341 \cdot 5 + 227$$
$$341 = 227 \cdot 1 + 114$$
$$227 = 114 \cdot 1 + 113$$
$$114 = 113 \cdot 1 + 1,$$

we have

$$\begin{aligned} 1 &= 114 - 113 \cdot 1 \\ &= 114 - (227 - 114 \cdot 1) \cdot 1 \\ &= 227 \cdot (-1) + 114 \cdot 2 \\ &= 227 \cdot (-1) + (341 - 227 \cdot 1) \cdot 2 \\ &= 341 \cdot 2 + 227 \cdot (-3) \\ &= 341 \cdot 2 + (1932 - 341 \cdot 5) \cdot (-3) \\ &= 1932 \cdot (-3) + 341 \cdot 17 \\ &= 1932 \cdot (-3) + (2273 - 1932 \cdot 1) \cdot 17 \\ &= 2273 \cdot 17 - 1932 \cdot 20. \end{aligned}$$

Hence, the pair of $a = 17$ and $b = 20$ is one pair of integer solution.
Since 5 and $\varphi(2021)$ $(= 1932)$ are relatively prime, we have

$$5^{2273a} = 5^{1932b+1}$$
$$(5^a)^{2273} = (5^{\varphi(2021)})^b \cdot 5$$
$$(5^a)^{2273} \equiv 5 \pmod{2021}.$$

Since

$$\begin{aligned} 5^a &= 5^{10} \cdot 5^7 \\ &\equiv 153 \cdot 1327 \pmod{2021} \\ &\equiv 931 \pmod{2021}, \end{aligned}$$

we obtain

$$x = 931.$$

(Answer) $x = 931$

2

(1) ① $\Gamma(t+1)=\int_0^{\infty} x^t e^{-x}dx=\left[-x^t e^{-x}\right]_0^{\infty}+t\int_0^{\infty} x^{t-1}e^{-x}dx=t\Gamma(t).$

② Since $\Gamma(1)=\int_0^{\infty} e^{-x}dx=\left[-e^{-x}\right]_0^{\infty}=1$,

using ① for a positive integer n, we have

$$\Gamma(n+1)=n\Gamma(n)=n(n-1)\Gamma(n-1)=\ldots=n!\Gamma(1)=n!.$$

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(2) Letting $y=\log_e \frac{1}{t}$, since $t=e^{-y}$ and $\frac{dt}{dy}=-e^{-y}$, we have

t	$0\to 1$
y	$\infty\to 0$

$$\int_0^1 t^3\left(\log_e \frac{1}{t}\right)^{\frac{5}{2}}dt=\int_{\infty}^0 e^{-3y}y^{\frac{5}{2}}(-e^{-y})dy=\int_0^{\infty} y^{\frac{5}{2}}e^{-4y}dy.$$

Letting $x=4y$, we have $y=\frac{x}{4}$ and $\frac{dy}{dx}=\frac{1}{4}$. Using the result of ① in (1), we have

y	$0\to\infty$
x	$0\to\infty$

$$\begin{aligned}\int_0^{\infty} y^{\frac{5}{2}}e^{-4y}dy&=\int_0^{\infty}\left(\frac{x}{4}\right)^{\frac{5}{2}}e^{-x}\cdot\frac{1}{4}dx\\&=\frac{1}{128}\int_0^{\infty} x^{\frac{7}{2}-1}e^{-x}dx\\&=\frac{1}{128}\Gamma\left(\frac{7}{2}\right)\\&=\frac{1}{128}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right).\end{aligned}$$

For

$$\Gamma\left(\frac{1}{2}\right)=\int_0^{\infty} x^{-\frac{1}{2}}e^{-x}dx,$$

x	$0\to\infty$
z	$0\to\infty$

letting $x=z^2$, since $\frac{dx}{dz}=2z$, we have

$$\Gamma\left(\frac{1}{2}\right)=\int_0^{\infty} z^{-1}e^{-z^2}\cdot 2zdz=2\int_0^{\infty} e^{-z^2}dz=\sqrt{\pi}.$$

Therefore, we obtain

$$\int_0^1 t^3\left(\log_e \frac{1}{t}\right)^{\frac{5}{2}}dt=\frac{1}{128}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}=\frac{15\sqrt{\pi}}{1024}.$$

(Answer) $\frac{15\sqrt{\pi}}{1024}$

3

Let

$$\boldsymbol{p}=\begin{pmatrix}\dfrac{a}{\sqrt{3+2\sqrt{3}\sin A}}\\ \dfrac{b}{\sqrt{3+2\sqrt{3}\sin B}}\\ \dfrac{c}{\sqrt{3+2\sqrt{3}\sin C}}\end{pmatrix} \quad \text{and} \quad \boldsymbol{q}=\begin{pmatrix}\sqrt{3+2\sqrt{3}\sin A}\\ \sqrt{3+2\sqrt{3}\sin B}\\ \sqrt{3+2\sqrt{3}\sin C}\end{pmatrix}.$$

Since $-1\le\dfrac{\boldsymbol{p}\cdot\boldsymbol{q}}{|\boldsymbol{p}|\,|\boldsymbol{q}|}\le 1$ for the inner product $\boldsymbol{p}\cdot\boldsymbol{q}$, we have

$$|\boldsymbol{p}|^2\,|\boldsymbol{q}|^2\ge(\boldsymbol{p}\cdot\boldsymbol{q})^2.$$

Since

$$(\boldsymbol{p}\cdot\boldsymbol{q})^2=(a+b+c)^2=9,$$

$$|\boldsymbol{p}|^2=T,$$

$$|\boldsymbol{q}|^2=9+2\sqrt{3}(\sin A+\sin B+\sin C),$$

we have

$$T\ge\frac{9}{9+2\sqrt{3}(\sin A+\sin B+\sin C)}.$$

Here, for $f(x)=\sin x$, since $f'(x)=\cos x$ and $f''(x)=-\sin x$, $f''(x)<0$ for $0<x<\pi$, which implies that the function $f(x)$ is concave down.
Since the values of A, B and C are all positive real numbers less than π, by Jensen's inequality, we have

$$\sin A+\sin B+\sin C\le 3\sin\frac{A+B+C}{3}=3\sin\frac{\pi}{3}=\frac{3\sqrt{3}}{2}.$$

It follows that

$$\frac{9}{9+2\sqrt{3}(\sin A+\sin B+\sin C)}\ge\frac{9}{9+2\sqrt{3}\cdot\dfrac{3\sqrt{3}}{2}}=\frac{1}{2}.$$

Therefore, we obtain $T\ge\dfrac{1}{2}$.

For equilateral triangle △ABC with sides of length 1, since

$$a=b=c=1,\quad A=B=C=\frac{\pi}{3}$$

and $\sin A=\sin B=\sin C=\dfrac{\sqrt{3}}{2}$, the value of T is

$$\frac{1}{3+2\sqrt{3}\cdot\dfrac{\sqrt{3}}{2}}\cdot 3=\frac{1}{2}.$$

Therefore, the minimum value of T is $\dfrac{1}{2}$.

(Answer) $\dfrac{1}{2}$

4

From the result, the number of strikes is 30% of the total and the number of spares is 20% of the total. Hence, the table shows the expected numbers of strikes and spares.
Note that the test statistic follows, approximately, a χ^2-distribution with degrees of freedom $(3-1)\cdot(3-1)=4$.
The test value is

	A	B	C	Total
Strike	30	36	24	90
Spare	20	24	16	60
Other	50	60	40	150
Total	100	120	80	300

$$T=\frac{(27-30)^2}{30}+\frac{(45-36)^2}{36}+\frac{(18-24)^2}{24}+\frac{(18-20)^2}{20}+\frac{(18-24)^2}{24}$$
$$+\frac{(24-16)^2}{16}+\frac{(55-50)^2}{50}+\frac{(57-60)^2}{60}+\frac{(38-40)^2}{40}$$
$$=10.5.$$

Using the 0.05 column and the row corresponding to the degrees of freedom 4, we have 9.4877, which is less than the value of T. Therefore, T is in the rejection region. It follows that H_0 is rejected.

(Answer) H_0 is rejected

5

(1) The set of all positive integers **N** is divided into the following three sets.

$A = \{ 2k \mid k$ is a positive integer $\} = \{ 2, 4, 6, 8, 10, 12, \dots \}$,
$B = \{ 4k+1 \mid k$ is an integer greater than or equal to $0 \} = \{ 1, 5, 9, \dots \}$,
$C = \{ 4k+3 \mid k$ is an integer greater than or equal to $0 \} = \{ 3, 7, 11, \dots \}$.

(i) For m that is an element of A, from (Ⅰ), if $m = 2$, then $n = 3$. It follows that as m is increased by 2, $m = 4, 6, 8, \dots$, n is increased by 3. Hence, the possible values of $n = f(m)$ are all positive integers that are divisible by 3.

(ii) For m that is an element of B, from (Ⅱ), if $m = 1$, then $n = 1$. It follows that as m is increased by 4, $m = 5, 9, 13, \dots$, n is increased by 3. Hence, the possible values of $n = f(m)$ are all positive integers that leave a remainder of 1 when divided by 3.

(iii) For m that is an element of C, from (Ⅲ), if $m = 3$, then $n = 2$. It follows that as m is increased by 4, $m = 7, 11, 15, \dots$, n is increased by 3. Hence, the possible values of $n = f(m)$ are all positive integers that leave a remainder of 2 when divided by 3.

From (i), (ii) and (iii), since there exists a positive integer m such that $n = f(m)$ for all positive integers n, the mapping f is surjective from **N** to **N**.

Next, for distinct two positive integers a_1 and a_2, if a_1 and a_2 are elements of distinct sets in A, B and C, by definition of mapping f, the remainders when dividing each of $f(a_1)$ and $f(a_2)$ by 3 are distinct. It follows that $f(a_1) \neq f(a_2)$.
If a_1 and a_2 are elements of set A, letting $a_1 = 2k_1$ and $a_2 = 2k_2$ gives $k_1 \neq k_2$. Hence, $3k_1 \neq 3k_2$, that is, $f(a_1) \neq f(a_2)$. In the same way, if a_1 and a_2 are elements of set B, we have $f(a_1) \neq f(a_2)$. In the same way, if a_1 and a_2 are elements of set C, we have $f(a_1) \neq f(a_2)$.
Hence, the mapping f is injective from **N** to **N**.

Therefore, the mapping f is bijective from **N** to **N**.

The inverse mappings, denoted by $m = f^{-1}(n)$, of $n = f(m)$ for an integer k' are as follows:

(Ⅰ′) If $n = 3k'$, $m = f^{-1}(n) = 2k'$.

(Ⅱ′) If $n = 3k'+1$, $m = f^{-1}(n) = 4k'+1$.

(Ⅲ′) If $n = 3k'+2$, $m = f^{-1}(n) = 4k'+3$.

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(2) (Example Answer) Starting with the number 2, the period is 2.
Starting with the number 4, the period is 5.
Starting with the number 44, the period is 12.

6	The determinant of $tI - A$, denoted by $\det(tI - A)$, is given by $$\det(tI-A) = \begin{vmatrix} t+3 & 3 & 5 \\ -3 & t-3 & -7 \\ -1 & -1 & t-1 \end{vmatrix}$$ $$= (t+3)(t-3)(t-1) + 21 + 15 - \{7(t+3) - 9(t-1) - 5(t-3)\}$$ $$= t^3 - t^2 - 2t.$$ By the Cayley-Hamilton theorem, we have $$A^3 - A^2 - 2A = O, \quad ①$$ where O is the 3×3 zero matrix. Letting $Q(x)$ and $ax^2 + bx + c$ be the quotient and remainder, respectively, when x^n is divided by $x^3 - x^2 - 2x$, where n is a positive integer and a, b and c are real numbers, we have $$x^n = (x^3 - x^2 - 2x)Q(x) + ax^2 + bx + c. \quad ②$$ Solving the cubic equation $x^3 - x^2 - 2x = 0$ gives $$x(x+1)(x-2) = 0$$ $$x = -1,\ 0,\ 2.$$ Substituting $x = -1,\ 0,\ 2$, respectively into both sides of ② gives $$\begin{cases} (-1)^n = a - b + c, & ③ \\ 0 = c, & ④ \\ 2^n = 4a + 2b + c. & ⑤ \end{cases}$$ Solving ③, ④ and ⑤ gives $$a = \frac{2^{n-1} + (-1)^n}{3},\quad b = \frac{2^{n-1} - 2\cdot(-1)^n}{3} \text{ and } c = 0.$$ Hence, for a 3×3 square matrix B, we have $$A^n = (A^3 - A^2 - 2A)B + \frac{2^{n-1} + (-1)^n}{3}A^2 + \frac{2^{n-1} - 2\cdot(-1)^n}{3}A.$$ Using ①, we obtain $$A^n = \frac{2^{n-1} + (-1)^n}{3}A^2 + \frac{2^{n-1} - 2\cdot(-1)^n}{3}A.$$ Therefore, we have $$p_n = \frac{2^{n-1} + (-1)^n}{3},\quad q_n = \frac{2^{n-1} - 2\cdot(-1)^n}{3} \text{ and } r_n = 0.$$ (Answer) $p_n = \frac{2^{n-1} + (-1)^n}{3}$, $q_n = \frac{2^{n-1} - 2\cdot(-1)^n}{3}$ and $r_n = 0$

7

For the given differential equation

$$y'' + 6y' + 5y = 26\cos x + 25x , \quad ①$$

first, we solve the corresponding homogeneous equation

$$y'' + 6y' + 5y = 0 . \quad ②$$

The characteristic equation here is $t^2 + 6t + 5 = 0$. Hence, we have

$$(t+1)(t+5) = 0$$
$$t = -5,\ -1.$$

Hence, the general solution of ② is

$$y = C_1 e^{-x} + C_2 e^{-5x},$$

where e is the base of the natural logarithm and C_1 and C_2 are arbitrary constants.
Next, letting $y = a\cos x + b\sin x + cx + d$, where a, b, c and d are constants, gives

$$y' = -a\sin x + b\cos x + c,$$
$$y'' = -a\cos x - b\sin x.$$

Substituting them into ① gives

$$(4a + 6b)\cos x + (-6a + 4b)\sin x + 5cx + 6c + 5d = 26\cos x + 25x.$$

Equating the corresponding coefficients, we have

$$\begin{cases} 4a + 6b = 26, & ③ \\ -6a + 4b = 0, & ④ \\ 5c = 25, & ⑤ \\ 6c + 5d = 0. & ⑥ \end{cases}$$

Solving the system of equations ③, ④, ⑤ and ⑥ gives

$$a = 2,\ b = 3,\ c = 5 \text{ and } d = -6 .$$

Hence, the general solution of the given differential equation is

$$y = C_1 e^{-x} + C_2 e^{-5x} + 2\cos x + 3\sin x + 5x - 6.$$

Since $y' = -C_1 e^{-x} - 5C_2 e^{-5x} - 2\sin x + 3\cos x + 5$, $y(0) = -4$ and $y'(0) = 12$, we have

$$\begin{cases} C_1 + C_2 + 2 - 6 = -4, & ⑦ \\ -C_1 - 5C_2 + 3 + 5 = 12. & ⑧ \end{cases}$$

Solving the system of equations ⑦ and ⑧ gives

$$C_1 = 1 \text{ and } C_2 = -1.$$

Therefore, the solution of the differential equation is

$$y = e^{-x} - e^{-5x} + 2\cos x + 3\sin x + 5x - 6.$$

(Answer) $y = e^{-x} - e^{-5x} + 2\cos x + 3\sin x + 5x - 6$