Pre 1st Kyu

Section 2：Application Test

Test Time : 120 minutes

───── Test Instructions ─────

1．Make sure that you have the correct level (Kyu) test.

2．Do not open the booklet until you are told to do so.

3．Write your name and examinee number on this page.

4．Write your name, examinee number and other necessary information on the answer sheets.

5．Write your answers on the answer sheets（they are numbered 1 through 4）. Write the steps leading to your answer. However if there are specific instructions for a problem, follow the instructions.

6．Problems 1 to 5 are selective problems. Choose two problems from the selective problems and fill in 〇 to indicate which problems you chose. Then write your answers. Note that all of your answers will not be marked if you answered more than two problems from the selective problems. Problems 6 and 7 are required problems.

7．You may use a calculator.

8．Turn off your cell phone and do not use it during the test.

9．Ask an examination supervisor if your problem sheets have inconsistent page numbering or missing pages.

10. It is prohibited to disclose the problems to the general public, such as on the Internet, without permission.

Name	
Examinee Number	－

[Pre-1st Kyu]   Section 2: Application Test

1   (Selective)

Which number is greater, $\sin(\alpha + \beta + \gamma)$ or $\cos(\alpha + \beta + \gamma)$ if

$$\tan\alpha = \frac{1}{2}, \quad \tan\beta = \frac{1}{5} \quad \text{and} \quad \tan\gamma = \frac{1}{8}$$

for $0 < \alpha < \dfrac{\pi}{2}$, $0 < \beta < \dfrac{\pi}{2}$ and $0 < \gamma < \dfrac{\pi}{2}$?

2   (Selective)

For any positive integer n, the following equalities hold (you don't need to prove them).

$$\sum_{k=1}^{n} k = \frac{1}{2}n(n+1), \quad \sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1), \quad \sum_{k=1}^{n} k^3 = \left\{\frac{1}{2}n(n+1)\right\}^2$$

S_n is the sum of the first n terms of a sequence $\{a_n\}$ and is given by

$$S_n = \left\{\frac{1}{6}n(n+1)(2n+1)\right\}^2. \qquad \qquad \textit{(Expression skill)}$$

(1) a_n is the nth term of the sequence $\{a_n\}$ and its degree of the leading term is quintic (polynomial of degree 5). Find the quintic expression and write it in expanded form.

(2) $\displaystyle\sum_{k=1}^{n} k^5$ is expressed as a polynomial of degree 6. Find this polynomial of degree 6 using the result of (1) and write it in factored form.

3   (Selective)

In this problem, $\overline{\text{AB}}$ represents the length of line segment AB.

For the curve $y = 2\sqrt{x}$ ($x \geq 0$) in the xy-plane with the origin O, point P(t, $2\sqrt{t}$) lies on the curve and point Q lies on the y-axis such that $\overline{\text{OP}} = \overline{\text{OQ}}$, where $t > 0$ and the y-coordinate of the point Q is positive.

Let R be the point of intersection of the line PQ and the x-axis. Evaluate the following limit.

$$\lim_{t \to +0} \overline{\text{OR}}$$

4 (Selective)

2×2 square matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with real elements satisfies $A^2 = O$. Find the necessary and sufficient condition for a real number k such that 2×2 square matrix $kE - A$ has its inverse. Note that O represents the null matrix and E represents the identity matrix.

5 (Selective)

The square of a prime number greater than 3 is a number that leaves remainder 1 when divided by 3, such as $5^2 = 25$, $7^2 = 49$, $11^2 = 121$ and so on. Consider the following proposition (this proposition contains the fact above as a special case).

"The square of a prime number greater than 3 is a number that leaves remainder 1 when divided by an integer n."

Find the maximum positive integer n such that the proposition is true. Answer with your reasoning.

6 (Required)

Under certain conditions, it is known that the cooling speed of an object is proportional to the temperature difference between the object and its surrounding air.

When a cup of tea with a temperature of 80℃ is placed in a 20℃ room, the temperature of the tea T ℃ after t minutes is given by

$$T - 20 = C \cdot 10^{-kt},$$

where C and k are positive constants. Note that t is not always an integer.

(1) Find the value of the constant C.

(2) 22 minutes after placing the tea in the room, the temperature of the tea reached exactly 50℃. After placing the tea in the room, how many minutes will it take for the tea to reach exactly 25℃? Round your answer off to the nearest integer.
Use the values $\log_{10} 2 = 0.3010$ and $\log_{10} 3 = 0.4771$, if necessary.

7 (Required)

Consider the curve $y = \dfrac{x}{\sqrt{1+x^2}}$ for $0 \le x \le 1$ in the xy-plane. (*Measurement skill*)

(1) Find the volume V_1 of the solid formed by revolving the region bounded by the curve, the x-axis and the line $x = 1$ about the x-axis.

(2) Find the volume V_2 of the solid formed by revolving the region bounded by the curve, the y-axis and the line $y = \dfrac{1}{\sqrt{2}}$ about the y-axis.