1

Since $0<\alpha<\frac{\pi}{2}$ and $0<\tan\alpha<\frac{1}{\sqrt{3}}$, we have

$0<\alpha<\frac{\pi}{6}$.

Similarly, $0<\beta<\frac{\pi}{6}$ and $0<\gamma<\frac{\pi}{6}$.

Thus,

$0<\alpha+\beta+\gamma<\frac{\pi}{2}$.

We consider the value of $\tan(\alpha+\beta+\gamma)$. Using the trigonometric addition formula, we have

$$\tan(\alpha+\beta+\gamma)$$

$$=\frac{\tan(\alpha+\beta)+\tan\gamma}{1-\tan(\alpha+\beta)\tan\gamma}$$

$$=\frac{\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}+\tan\gamma}{1-\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}\times\tan\gamma}$$

$$=\frac{\tan\alpha+\tan\beta+(1-\tan\alpha\tan\beta)\tan\gamma}{1-\tan\alpha\tan\beta-(\tan\alpha+\tan\beta)\tan\gamma}$$

$$=\frac{\tan\alpha+\tan\beta+\tan\gamma-\tan\alpha\tan\beta\tan\gamma}{1-\tan\alpha\tan\beta-\tan\beta\tan\gamma-\tan\gamma\tan\alpha}.$$

Substituting

$$\tan\alpha=\frac{1}{2},\quad \tan\beta=\frac{1}{5}\quad\text{and}\quad \tan\gamma=\frac{1}{8}$$

into the formula, we have

$$\tan(\alpha+\beta+\gamma)$$

$$=\frac{\frac{1}{2}+\frac{1}{5}+\frac{1}{8}-\frac{1}{2}\cdot\frac{1}{5}\cdot\frac{1}{8}}{1-\frac{1}{2}\cdot\frac{1}{5}-\frac{1}{5}\cdot\frac{1}{8}-\frac{1}{8}\cdot\frac{1}{2}}$$

$$=\frac{40+16+10-1}{80-8-2-5}$$

$$=\frac{65}{65}$$

$$=1$$

Therefore, $\sin(\alpha+\beta+\gamma)=\cos(\alpha+\beta+\gamma)$.

(Answer) $\sin(\alpha+\beta+\gamma)=\cos(\alpha+\beta+\gamma)$ (Both are equal)

2

(1) $a_1=S_1=1$. If $n\geq 2$, we have

$$a_n=S_n-S_{n-1}$$

$$=\left\{\frac{1}{6}n(n+1)(2n+1)\right\}^2-\left\{\frac{1}{6}(n-1)n(2n-1)\right\}^2$$

$$=\frac{n^2}{36}\{(2n^2+3n+1)^2-(2n^2-3n+1)^2\}$$

$$=\frac{n^2}{36}\{(4n^2+2)\times 6n\}$$

$$=\frac{2}{3}n^5+\frac{1}{3}n^3$$

This is also true when $n=1$.

(Answer) $a_n=\frac{2}{3}n^5+\frac{1}{3}n^3$

(2) Using the result in (1),

$$\sum_{k=1}^{n}\frac{2k^5+k^3}{3}=S_n.$$

Hence

$$\sum_{k=1}^{n}k^5=\frac{3}{2}S_n-\frac{1}{2}\sum_{k=1}^{n}k^3$$

$$=\frac{3}{2}\left\{\frac{1}{6}n(n+1)(2n+1)\right\}^2-\frac{1}{2}\left\{\frac{1}{2}n(n+1)\right\}^2$$

$$=\frac{1}{24}n^2(n+1)^2\{(2n+1)^2-3\}$$

$$=\frac{1}{12}n^2(n+1)^2(2n^2+2n-1)$$

(Answer) $\sum_{k=1}^{n}k^5=\frac{1}{12}n^2(n+1)^2(2n^2+2n-1)$

3

Since the coordinates of P are $(t,\ 2\sqrt{t}\,)$ $(t>0)$,

$$\overline{OP}=\sqrt{t^2+(2\sqrt{t})^2}=\sqrt{t^2+4t}.$$

Thus the coordinates of Q are $(0,\ \sqrt{t^2+4t}\,)$. The slope of line PQ is

$$\frac{2\sqrt{t}-\sqrt{t^2+4t}}{t-0}=\frac{2\sqrt{t}-\sqrt{t^2+4t}}{t},$$

and the y-intercept is $\sqrt{t^2+4t}$.

Thus the equation of line PQ is

$$y=\frac{2\sqrt{t}-\sqrt{t^2+4t}}{t}x+\sqrt{t^2+4t}$$

Solving $0=\dfrac{2\sqrt{t}-\sqrt{t^2+4t}}{t}x+\sqrt{t^2+4t}$,

the x-coordinate of R is

$$x=\frac{t\sqrt{t^2+4t}}{\sqrt{t^2+4t}-2\sqrt{t}}.$$

Therefore,

$$\lim_{t\to+0}\overline{OR}$$

$$=\lim_{t\to+0}\frac{t\sqrt{t^2+4t}}{\sqrt{t^2+4t}-2\sqrt{t}}$$

$$=\lim_{t\to+0}\frac{t\sqrt{t^2+4t}(\sqrt{t^2+4t}+2\sqrt{t})}{t^2+4t-4t}$$

$$=\lim_{t\to+0}\frac{t(t^2+4t+2t\sqrt{t+4})}{t^2}$$

$$=\lim_{t\to+0}(t+4+2\sqrt{t+4})$$

$$=4+2\sqrt{4}$$

$$=8$$

(Answer) 8

4

Using the Cayley-Hamilton theorem for $A=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$, we have

$$A^2-(a+d)A+(ad-bc)E=O.\quad\cdots①$$

Substituting $A^2=O$ $\cdots②$ into ①,

$$(a+d)A=(ad-bc)E.\quad\cdots③$$

Letting $a+d\neq 0$, ③ can be expressed as $A=tE$ (t is a real number). Substituting this into ②, we have $t^2=0$, i.e. $t=0$. Thus $A=O$. This leads a contradiction (we supposed $a+d\neq 0$).

Thus, $a+d=0$. $\cdots④$

Substituting this into ③, $ad-bc=0$. $\cdots⑤$

The necessary and sufficient condition that $kE-A=\begin{pmatrix}k-a & -b\\ -c & k-d\end{pmatrix}$ has its inverse is that

$$(k-a)(k-d)-(-b)(-c)\neq 0,$$

that is,

$$k^2-(a+d)k+ad-bc\neq 0.\quad\cdots⑥$$

From ④, ⑤ and ⑥, the required necessary and sufficient condition is

$$k^2\neq 0.$$

Thus, $k\neq 0$.

(Answer) $k\neq 0$

5

Since $n\leq 5^2-1$, if we prove the proposition for $n=24$, this is the maximum value.

The remainder when a prime number greater than 3 is divided by 12 is one of

1, 5, 7 and 11.

Thus, a prime number greater than 3 can be expressed as either

$12k\pm 1$ or $12k\pm 5$ (k is an integer)

$$(12k\pm 1)^2$$

$$=144k^2\pm 24k+1$$

$$=24(6k^2\pm k)+1$$

(Signs correspond respectively)

Since $6k^2\pm k$ is an integer, the remainder when $(12k\pm 1)^2$ is divided by 24 is 1.

$$(12k\pm 5)^2$$

$$=144k^2\pm 120k+25$$

$$=24(6k^2\pm 5k+1)+1$$

(Signs correspond respectively)

Since $6k^2\pm 5k+1$ is an integer, the remainder when $(12k\pm 5)^2$ is divided by 24 is also 1.

Thus, the remainder when the square of a prime number greater than 3 is divided by 24 is 1.

Therefore, the required maximum value is 24.

(Answer) 24

6	(1) Since $T=80$ when $t=0$ for $T-20=C\cdot 10^{-kt}$, …① $80-20=C$. Hence we have $C=60$ and ① becomes $T-20=60\cdot 10^{-kt}$. …①′ (Answer) $C=60$ (2) Suppose that t_0 minutes after placing the tea in the room, the temperature reaches exactly 25℃. From ①′, we have $25-20=60\cdot 10^{-kt_0}$, so that $10^{kt_0}=\dfrac{60}{5}=12$. Therefore we get $t_0=\dfrac{\log_{10}12}{k}$. …②	Since $T=50$ when $t=22$, from ①′ we have $50-20=60\cdot 10^{-22k}$. Thus, $10^{22k}=\dfrac{60}{30}=2$. Therefore we have $k=\dfrac{\log_{10}2}{22}$. Substituting this into ②, $t_0=22\times\dfrac{\log_{10}12}{\log_{10}2}$ $=22\times\dfrac{2\log_{10}2+\log_{10}3}{\log_{10}2}$ $=22\times\dfrac{2\times0.3010+0.4771}{0.3010}$ $=78.8\cdots$ Therefore it takes 79 minutes. (Answer) 79 minutes
7	(1) For $y=\dfrac{x}{\sqrt{1+x^2}}$ $(0\le x\le 1)$, …① $y=0$ only if $x=0$. The required volume is $V_1=\pi\int_0^1 y^2\,dx=\pi\int_0^1\dfrac{x^2}{1+x^2}dx$ $=\pi\int_0^1\left(1-\dfrac{1}{1+x^2}\right)dx=\pi-\pi\int_0^1\dfrac{1}{1+x^2}dx$. Let $x=\tan\theta$, so that $\dfrac{dx}{d\theta}=\dfrac{1}{\cos^2\theta}$. Note that $0\to\dfrac{\pi}{4}$ for θ as $0\to1$ for x. Thus, we have $\int_0^1\dfrac{1}{1+x^2}dx=\int_0^{\frac{\pi}{4}}\dfrac{1}{1+\tan^2\theta}\cdot\dfrac{1}{\cos^2\theta}d\theta$ $=\int_0^{\frac{\pi}{4}}1\,d\theta=\dfrac{\pi}{4}$. Therefore, $V_1=\pi-\dfrac{\pi^2}{4}$. (Answer) $V_1=\pi-\dfrac{\pi^2}{4}$	(2) For ①, $y=0$ at $x=0$. Since $y^2=\dfrac{x^2}{1+x^2}$, we have $x^2=\dfrac{y^2}{1-y^2}$. Hence the required volume is $V_2=\pi\int_0^{\frac{1}{\sqrt2}}x^2dy=\pi\int_0^{\frac{1}{\sqrt2}}\dfrac{y^2}{1-y^2}dy$ $=-\pi\int_0^{\frac{1}{\sqrt2}}\left(1+\dfrac{1}{y^2-1}\right)dy$ $=-\dfrac{\pi}{\sqrt2}-\pi\int_0^{\frac{1}{\sqrt2}}\dfrac{1}{y^2-1}dy$. Since $\dfrac{1}{y^2-1}=\dfrac{1}{2}\left(\dfrac{1}{y-1}-\dfrac{1}{y+1}\right)$, $\int_0^{\frac{1}{\sqrt2}}\dfrac{1}{y^2-1}dy=\dfrac{1}{2}\int_0^{\frac{1}{\sqrt2}}\left(\dfrac{1}{y-1}-\dfrac{1}{y+1}\right)dy$ $=\dfrac{1}{2}\left[\log_e\dfrac{\lvert y-1\rvert}{\lvert y+1\rvert}\right]_0^{\frac{1}{\sqrt2}}=\dfrac{1}{2}\log_e\dfrac{\left\lvert\frac{1}{\sqrt2}-1\right\rvert}{\left\lvert\frac{1}{\sqrt2}+1\right\rvert}$ $=\dfrac{1}{2}\log_e\dfrac{\sqrt2-1}{\sqrt2+1}$ $=\log_e(\sqrt2-1)=-\log_e(\sqrt2+1)$. Therefore, $V_2=-\dfrac{\pi}{\sqrt2}+\pi\log_e(\sqrt2+1)$ (e is the base of the natural logarithm) (Answer) $V_2=\pi\log_e(\sqrt2+1)-\dfrac{\pi}{\sqrt2}$