1

There are no integer solutions $(x,\ y)$ for

$7x^2-9y^2=391.$ …①

We prove it as follows.

Since $391=17\times 23$, we take 17 as modulo. For $-8\leq n\leq 8$, the values of $7n^2$ and $9n^2 \pmod{17}$ are as follows.

n	$7n^2$	$9n^2$
0	0	0
±1	7	−8
±2	−6	2
±3	−5	−4
±4	−7	8
±5	5	4
±6	−3	1
±7	3	−1
±8	6	−2

If there are integer solutions $(x,\ y)$ that satisfy ①, we have

$$7x^2-9y^2\equiv 0 \pmod{17}.$$

From the table on the left, the only possibility is

$$x\equiv 0 \pmod{17} \text{ and } y\equiv 0 \pmod{17}.$$

Then we can express them as $x=17p$ and $y=17q$ (p and q are integers). Substituting them into ①, we obtain

$$7(17p)^2-9(17q)^2=391$$

$$7\cdot 17p^2-9\cdot 17q^2=23$$

$$7p^2-9q^2=\frac{23}{17}.$$

This leads a contradiction (the initial condition that p and q are integers cannot hold).

Therefore, there are no integer solutions $(x,\ y)$ for ①.

2

(1) If $x\neq 0$, $f(x)$ is differentiable.

$$f'(x)=\frac{x\cos x-\sin x}{x^2}.$$

This is continuous for $x\neq 0$.

For the derivative at $x=0$, we have

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x}=\lim_{x\to 0}\frac{1}{x}\left(\frac{\sin x}{x}-1\right)$$

$$=\lim_{x\to 0}\frac{\sin x-x}{x^2}=\lim_{x\to 0}\frac{(\sin x-x)'}{(x^2)'}$$

$$=\lim_{x\to 0}\frac{\cos x-1}{2x}=\lim_{x\to 0}\frac{(\cos x-1)'}{(2x)'}$$

$$=\lim_{x\to 0}\frac{-\sin x}{2}=0.$$

Hence, $f(x)$ is differentiable at $x=0$ as well and $f'(0)=0$. For the continuity at $x=0$,

$$\lim_{x\to 0}\frac{x\cos x-\sin x}{x^2}=\lim_{x\to 0}\frac{(x\cos x-\sin x)'}{(x^2)'}$$

$$=\lim_{x\to 0}\frac{\cos x-x\sin x-\cos x}{2x}=\lim_{x\to 0}\frac{-\sin x}{2}$$

$$=0=f'(0)$$

Therefore $f(x)$ is differentiable for all real numbers x and its derivative function $f'(x)$ is continuous.

(2) From the result in (1), we have

$$a_n=f'(n\pi)=\frac{n\pi\cos(n\pi)-\sin(n\pi)}{(n\pi)^2}$$

$$=\frac{n\pi(-1)^n}{(n\pi)^2}=\frac{(-1)^n}{n\pi}\quad (n\geq 1),$$

so that

$$\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{\infty}\frac{(-1)^n}{n\pi}=-\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}.$$

The Maclaurin series expansion for $\log_e(1+x)$ is

$$\log_e(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\cdots$$

$$=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^n,$$ (e is the base of the natural logarithm)

which can hold for $-1<x\leq 1$. Letting $x=1$, we obtain

$$\log_e 2=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}.$$

Therefore, the sum of the series is

$$\sum_{n=1}^{\infty}a_n=-\frac{1}{\pi}\log_e 2$$

(Answer) $-\dfrac{1}{\pi}\log_e 2$

3

(1) Let $M=\begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix}$. Then, the column vectors of M are orthogonal one another and their magnitudes are all ℓ. Letting ${}^tM$ be the transpose of M and E be the 3×3 identity matrix, we have

$$M\,{}^tM=\ell^2E.$$

Hence, $\frac{1}{\ell}\,{}^tM$ is the inverse of $\frac{1}{\ell}M$, i.e.,

$$\left(\frac{1}{\ell}\,{}^tM\right)\left(\frac{1}{\ell}M\right)=E.$$

Therefore, we obtain

$$x_1^2+x_2^2+x_3^2=\ell^2 \quad \cdots①$$

$$y_1^2+y_2^2+y_3^2=\ell^2 \quad \cdots②$$

$$x_1y_1+x_2y_2+x_3y_3=0 \quad \cdots③$$

From ①－②＋$2i$×③,

$$(x_1+iy_1)^2+(x_2+iy_2)^2+(x_3+iy_3)^2=0,$$

that is, the following equality always holds

$$\alpha^2+\beta^2+\gamma^2=0.$$

(Answer) $\alpha^2+\beta^2+\gamma^2=0$

(2) From ①＋②,

$$2\ell^2$$

$$=(x_1^2+y_1^2)+(x_2^2+y_2^2)+(x_3^2+y_3^2)$$

$$=|\alpha|^2+|\beta|^2+|\gamma|^2$$

Therefore,

$$\ell=\sqrt{\frac{1}{2}(|\alpha|^2+|\beta|^2+|\gamma|^2)}$$

(Answer) $\ell=\sqrt{\frac{1}{2}(|\alpha|^2+|\beta|^2+|\gamma|^2)}$

4

(1) Let x_k g $(k=1,\ 2,\ 3,\ \cdots,\ 10)$ be the given weights. For $n=10$, we have

$$\bar{x}=15.00 \quad \text{and} \quad \sum_{k=1}^{n}(x_k-\bar{x})^2=0.2646.$$

Thus, the sample mean is 15.00 g and the sample standard deviation s is

$$s=\sqrt{\frac{0.2646}{10-1}} \text{ (g)}.$$

Thus, $X=\dfrac{15.00-m}{\frac{s}{\sqrt{10}}}$ follows the t-distribution with 9 degrees of freedom. For the t-distribution with 9 degrees of freedom, the value of v that satisfies

$$P(-v\le X\le v)=0.95$$

is $v=2.262$ according to the table. Hence

$$-2.262\le X\le 2.262.$$

Solving it for m,

$$15.00-2.262\times\frac{s}{\sqrt{10}}$$

$$\le m\le 15.00+2.262\times\frac{s}{\sqrt{10}}.$$

Since $2.262\times\frac{s}{\sqrt{10}}=0.1226\cdots$,

$$15.00-2.262\times\frac{s}{\sqrt{10}}=14.877\cdots$$

$$15.00+2.262\times\frac{s}{\sqrt{10}}=15.122\cdots$$

Therefore, the confidence interval is

$$14.87\le m\le 15.13$$

(Answer) $14.87\le m\le 15.13$

(2) The sample mean is 15.00 g and the sample standard deviation is 0.250 g. Since the number of samples, 200, may be considered to be large enough, we may consider that $X=\dfrac{15.00-m}{\frac{0.250}{\sqrt{200}}}$ follows the t-distribution with ∞ degrees of freedom (this is equivalent to the normal distribution).
For the t-distribution with ∞ degrees of freedom, the value of w that satisfies

$$P(-w\le X\le w)=0.95$$

is $w=1.960$ according to the table. Hence

$$-1.960\le X\le 1.960.$$

Solving it for m,

$$15.00-1.960\times\frac{0.250}{\sqrt{200}}$$

$$\le m\le 15.00+1.960\times\frac{0.250}{\sqrt{200}}.$$

Since $1.960\times\frac{0.250}{\sqrt{200}}=0.0346\cdots$,

$$15.00-1.960\times\frac{0.250}{\sqrt{200}}=14.965\cdots$$

$$15.00+1.960\times\frac{0.250}{\sqrt{200}}=15.034\cdots$$

Therefore, the confidence interval is

$$14.96\le m\le 15.04$$

(Answer) $14.96\le m\le 15.04$

5

Let ΔP be the length of line segment PP'. Let TT' be the tangent line of C at P as shown in Figure 1. Letting $\theta = \angle\text{SPT}$, we have $\angle\text{S'PT'} = \theta$ (property of an ellipse).

From ①, the area of $\triangle$SPP' is the area of a triangle whose two lengths of sides are SP and ΔP, and the included angle is $\pi - \theta$ as shown in Figure 2. Hence, the area of $\triangle$SPP' can be expressed as

$$\frac{1}{2}\text{SP}\sin(\pi-\theta)\times\Delta\text{P}\ .$$

From the Kepler's 2nd law and $\sin(\pi-\theta)=\sin\theta$, let

$$\frac{1}{2}\text{SP}\sin\theta\times\frac{\Delta\text{P}}{\Delta t}=k\ , \qquad (*)$$

where k is a constant.

Further, letting $\Delta\phi = \angle\text{PS'P'}$, we have $\angle\text{S'PP'} \approx \theta$ from ①. Since Δt is small enough, $\Delta\phi$ is also small enough. Using the law of sines, we can express

$$\frac{\Delta\text{P}}{\Delta\phi}\approx\frac{\Delta\text{P}}{\sin\Delta\phi}\approx\frac{\text{S'P'}}{\sin\theta}\ ,$$

Shown in Figure 3.

Figure 1

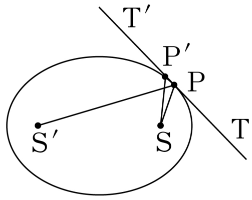


Figure 2

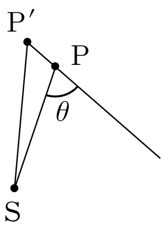
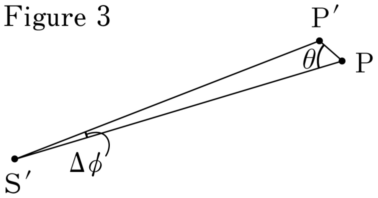


Figure 3



Using ②, we have

$$\frac{\dfrac{\Delta\text{P}}{\Delta t}}{\dfrac{\Delta\phi}{\Delta t}}\approx\frac{\Delta\text{P}}{\Delta\phi}\approx\frac{\text{S'P'}}{\sin\theta}\approx\frac{\text{S'P}}{\sin\theta}\ .$$

From this formula and (*), we have

$$\frac{\Delta\phi}{\Delta t}\approx\frac{\sin\theta}{\text{S'P}}\times\frac{\Delta\text{P}}{\Delta t}=\frac{2k}{\text{SP}\times\text{S'P}}\ . \qquad (**)$$

For the elliptic orbit C, we have

$$\text{SP}+\text{S'P}=2a\ ,$$

where $2a$ is the length of the major axis. Letting $\text{SP}=x$, $\text{S'P}=2a-x$. Thus, we have

$$\text{SP}\times\text{S'P}=x(2a-x)=-(x-a)^2+a^2\ .$$

Since

$$\left|\text{SP}-\text{S'P}\right|\le 2ae$$

for C, the range of x is $a-ae\le x\le a+ae$. Therefore, $\text{SP}\times\text{S'P}$ takes

Maximum value a^2 at $x=a$
Minimum value $a^2-a^2e^2$ at $x=a-ae$ or $x=a+ae$

From ③, we may assume $e^2\approx 0$, so that $\text{SP}\times\text{S'P}\approx a^2$. Therefore, (**) can be written as

$$\frac{\Delta\phi}{\Delta t}\approx\frac{2k}{a^2}\ ,$$

so that the value of $\dfrac{\Delta\phi}{\Delta t}$ is approximately constant. This means that we can regard the angular velocity of the planet around S' being constant.

6

(1) Let a_{ij} be the element at $(i,\ j)$ of A and Let b_{ij} be the element at $(i,\ j)$ of B. From the definition of the product of matrices, the element at $(p,\ p)$ of AB can be expressed as

$$\sum_{k=1}^{n} a_{pk}b_{kp}\ (1 \le p \le n).$$

Thus, we have

$$\mathrm{tr}\ (AB) = \sum_{p=1}^{n}\left(\sum_{k=1}^{n} a_{pk}b_{kp}\right).$$

Similarly, we get $\mathrm{tr}\ (BA) = \sum_{q=1}^{n}\left(\sum_{\ell=1}^{n} b_{q\ell}a_{\ell q}\right)$.

Since

$$\sum_{p=1}^{n}\left(\sum_{k=1}^{n} a_{pk}b_{kp}\right) = \sum_{p=1}^{n}\left(\sum_{k=1}^{n} b_{kp}a_{pk}\right) = \sum_{k=1}^{n}\left(\sum_{p=1}^{n} b_{kp}a_{pk}\right)$$

we obtain $\mathrm{tr}\ (AB) = \mathrm{tr}\ (BA)$.

(2) From the linearity of tr and (1),

$$\mathrm{tr}(A) = \mathrm{tr}(AB - BA) = \mathrm{tr}(AB) - \mathrm{tr}(BA) = 0.$$

For an integer m greater than or equal to 2,

$$A^m = A^{m-1}(AB - BA) = A(A^{m-1}B) - (A^{m-1}B)A.$$

Letting $A^{m-1}B = C$ and using (1),

$$\mathrm{tr}(A^m) = \mathrm{tr}(AC - CA) = \mathrm{tr}(AC) - \mathrm{tr}(CA) = 0.$$

Thus,

$$\mathrm{tr}(A) = \mathrm{tr}(A^2) = \cdots = \mathrm{tr}(A^n) = 0. \quad \cdots①$$

Let $\lambda_j\ (1 \le j \le n)$ be the eigenvalues of A (including the algebraic multiplicity), the eigenvalue of A^m (m is a positive integer) is $\lambda_j{}^m\ (1 \le j \le n)$. Since the sum of diagonal elements is equal to the sum of eigenvalues,

$$\mathrm{tr}(A^m) = \sum_{j=1}^{n} \lambda_j{}^m.$$

From this result and ①, every sum of eigenvalues of A to the m th power $(1 \le m \le n)$ is 0.

Therefore, every eigenvalue of A is 0 and the characteristic polynomial of A is x^n. Thus, from the Cayley-Hamilton theorem, $A^n = O$.

(Answer) $A^n = O$

7

Since $\dfrac{\partial^3 f}{\partial x \partial y \partial z} = 0$, f does not contain the xyz-term.

Since

$$\frac{\partial^3 f}{\partial x^3} = \frac{\partial^2}{\partial x^2}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\right) - \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial y \partial z} + \frac{\partial^2 f}{\partial z \partial x}\right) + \frac{\partial^3 f}{\partial x \partial y \partial z}$$

$$= 0 - 0 + 0 = 0,$$

f does not contain the x^3-term. Similarly, f does not contain the y^3- and z^3-terms. Thus, the form of f is

$$f = ax^2y + bxy^2 + cy^2z + dyz^2 + pz^2x + qzx^2,$$

where a, b, c, d, p and q are real numbers.

Since

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 - 2\left(\frac{\partial^2}{\partial x \partial y} + \frac{\partial^2}{\partial y \partial z} + \frac{\partial^2}{\partial z \partial x}\right),$$

We have

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$= 2ay + 2qz + 2bx + 2cz + 2dy + 2px$$

$$= 2(b+p)x + 2(a+d)y + 2(c+q)z.$$

Since this is equal to 0 for all real numbers x, y and z,

$$b + p = 0,\quad a + d = 0,\quad c + q = 0. \quad \cdots①$$

Further,

$$\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial y \partial z} + \frac{\partial^2 f}{\partial z \partial x}$$

$$= 2ax + 2by + 2cy + 2dz + 2pz + 2qx$$

$$= 2(a+q)x + 2(b+c)y + 2(d+p)z.$$

Since this is equal to 0,

$$a + q = 0,\quad b + c = 0,\quad d + p = 0. \quad \cdots②$$

From ① and ②,

$$a = c = p,\quad b = d = q,\quad a + b = 0.$$

Therefore, f is the constant multiple (except for 0) of

$$x^2y - xy^2 + y^2z - yz^2 + z^2x - zx^2$$

$$= -(x-y)(y-z)(z-x)$$

(Answer) Constant multiple of the elementary symmetric polynomial $(x-y)(y-z)(z-x)$.