[Pre-1st Kyu] Section 2: Application Test

1 (Selective)

Answer the following for $\triangle$ABC on a plane. (*Proof skill*)

(1) Prove

$$\sin A + \sin B \leq 2\cos\frac{C}{2}.$$

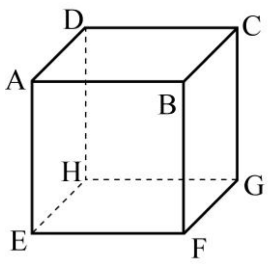
(2) Prove the following inequality and answer the type of triangle when the equality sign holds.

$$\sin A + \sin B + \sin C \leq \cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2}$$

2 (Selective)

The figure on the right shows a cube ABCD-EFGH of side length 1. Let α be the plane that passes through three points, B, D and E. Let β be the plane that passes through three points, C, F and H. Let P be the point of intersection of diagonal AG and the plane α. Let Q be the point of intersection of diagonal AG and the plane β Further let

$\overrightarrow{\mathrm{AB}} = \vec{b}$, $\overrightarrow{\mathrm{AD}} = \vec{d}$ and $\overrightarrow{\mathrm{AE}} = \vec{e}$.



(1) Express $\overrightarrow{\mathrm{PQ}}$ using $\vec{b}$, $\vec{d}$ and $\vec{e}$.

(2) Find the volume of the tetrahedron PCFH.

3 (Selective)

There is a hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, where $a > 0$ and $b > 0$, on xy-plane with the origin O. Let A be a point at (p, q) on the hyperbola. Let S and T be the points of intersection of the tangent line at point A and two asymptotes of the hyperbola. Show that the area of $\triangle$OST is a constant, independent of point A. (*Proof skill*)

4 (Selective)

Consider 2×2 matrix $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ that satisfies $A\neq \pm E$ and $A^4=E$, where a, b, c and d are real numbers and E represents the identity matrix.

(1) Find the value of $a+d$.

(2) Find the range of a when $bc\geq 0$.

5 (Selective)

Find all sets of integers, (x, y, z), satisfying the following equality.

$$x^2+y^2+z^2=2xyz$$

6 (Required)

Prove that if nonzero real numbers a, b and c satisfy the equality

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=1,$$

then $a+b=0$ or $b+c=0$ or $c+a=0$ must hold. (*Proof skill*)

7 (Required)

Consider the curve $y=\sqrt{x^2-1}$, where $x\geq 1$, on xy-plane with the origin O.

(1) There exists only one pair of constants a and b, (a, b), such that the following equality is satisfied for any real number t for $t\geq 1$. Find the pair of (a, b). Note that e represents the base of natural logarithms.

$$\int_1^t \sqrt{x^2-1}\,dx = at\sqrt{t^2-1}+b\log_e\left(t+\sqrt{t^2-1}\right)$$

(2) Take two distinct points A(1, 0) and B(p, $\sqrt{p^2-1}$) on the curve. Let $\frac{S}{2}$ be the area bounded by the curve, straight lines OA and OB. Express p in terms of S.

(*Expression skill*)