1

(1) $A+B+C=180°$ and

$$\sin A+\sin B=2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$
$$=2\sin\frac{180°-C}{2}\cos\frac{A-B}{2}$$
$$=2\cos\frac{C}{2}\cos\frac{A-B}{2}.$$

Since $\cos\frac{A-B}{2}\leq 1$ and $\cos\frac{C}{2}>0$, we have

$$\sin A+\sin B\leq 2\cos\frac{C}{2}\quad (\text{equal sign} \Leftrightarrow A=B)\quad \cdots①$$

(2) We replace the angles as follows (cyclic replacement)

$A\to B,\quad B\to C,\quad C\to A$

From the same argument in (1), we have

$$\sin B+\sin C\leq 2\cos\frac{A}{2}\quad (\text{equal sign} \Leftrightarrow B=C)\quad \cdots②$$

Similarly, for ② we replace $A\to B$, $B\to C$, $C\to A$.

$$\sin C+\sin A\leq 2\cos\frac{B}{2}\quad (\text{equal sign} \Leftrightarrow C=A)\quad \cdots③$$

Adding ①, ② and ③ up and dividing the sum by 2, we have

$$\sin A+\sin B+\sin C\leq \cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\quad \cdots④$$

The necessary and sufficient condition that ④ holds the equal sign is that each of ①, ② and ③ holds the equal sign. That is

$A=B$, $B=C$ and $C=A$

So, $A=B=C$. When ④ holds the equal sign, △ABC is an equilateral triangle.

(Answer) Equilateral triangle

2

(1) From the given conditions, we have

$$\overrightarrow{AC}=\vec{b}+\vec{d},\quad \overrightarrow{AF}=\vec{b}+\vec{e},$$
$$\overrightarrow{AH}=\vec{d}+\vec{e},\quad \overrightarrow{AG}=\vec{b}+\vec{d}+\vec{e}$$

Since point P lies on line segment AG,

$$\overrightarrow{AP}=k\overrightarrow{AG}=k(\vec{b}+\vec{d}+\vec{e})\ (0<k<1)\quad \cdots①$$

Also, point P lies on the surface α passing through three points B, D and E, $k+k+k=1$ $\therefore$ $k=\frac{1}{3}$

Hence, $\overrightarrow{AP}=\frac{1}{3}\overrightarrow{AG}$ $\cdots②$

Using the symmetry of the solid, $\overrightarrow{GQ}=-\overrightarrow{AP}$ $\cdots③$

From ② and ③, we get

$$\overrightarrow{AQ}=\overrightarrow{AG}+\overrightarrow{GQ}$$
$$=\overrightarrow{AG}-\overrightarrow{AP}=\frac{2}{3}\overrightarrow{AG}\quad \cdots④$$

From ② and ④, we get

$$\overrightarrow{PQ}=\overrightarrow{AQ}-\overrightarrow{AP}=\frac{1}{3}\overrightarrow{AG}=\frac{1}{3}(\vec{b}+\vec{d}+\vec{e})$$

(Answer) $\overrightarrow{PQ}=\frac{1}{3}(\vec{b}+\vec{d}+\vec{e})$

(2) Since △CFH is an equilateral triangle of side $\sqrt{2}$, the area is

$$\frac{1}{2}\times\sqrt{2}\times\sqrt{2}\times\frac{\sqrt{3}}{2}=\frac{\sqrt{3}}{2}\quad \cdots⑤$$

Since $|\vec{b}|=|\vec{d}|=|\vec{e}|=1$ and $\vec{b}\cdot\vec{d}=\vec{d}\cdot\vec{e}=\vec{e}\cdot\vec{b}=0$, we get

$$\overrightarrow{CF}\cdot\overrightarrow{AG}=(\overrightarrow{AF}-\overrightarrow{AC})\cdot\overrightarrow{AG}$$
$$=(\vec{e}-\vec{d})\cdot(\vec{b}+\vec{d}+\vec{e})=0$$
$$\overrightarrow{CH}\cdot\overrightarrow{AG}=(\overrightarrow{AH}-\overrightarrow{AC})\cdot\overrightarrow{AG}$$
$$=(\vec{e}-\vec{b})\cdot(\vec{b}+\vec{d}+\vec{e})=0$$

Hence, $\overrightarrow{AG}$ is a normal vector to the plane β. The height of tetrahedron PCFH when regarding △CFH as its base is $|\overrightarrow{PQ}|=\frac{1}{3}|\vec{b}+\vec{d}+\vec{e}|$. Since $|\vec{b}+\vec{d}+\vec{e}|^2=(\vec{b}+\vec{d}+\vec{e})\cdot(\vec{b}+\vec{d}+\vec{e})=3$, we have $|\overrightarrow{PQ}|=\frac{\sqrt{3}}{3}$. $\cdots⑥$ Therefore, from ⑤ and ⑥, the volume of tetrahedron PCFH is

$$\frac{1}{3}\times\frac{\sqrt{3}}{2}\times\frac{\sqrt{3}}{3}=\frac{1}{6}$$

(Answer) $\frac{1}{6}$

3

The equation of the tangent line at A $(p,\ q)$ is

$$\frac{px}{a^2}-\frac{qy}{b^2}=1 \quad \cdots ①$$

The coordinates of the point of intersection of ① and the asymptote $y=\frac{b}{a}x$ are given by

$$\frac{px}{a^2}-\frac{qx}{ab}=1,$$

$$x=\frac{a}{\frac{p}{a}-\frac{q}{b}},$$

Then $\left(\frac{a}{\frac{p}{a}-\frac{q}{b}},\frac{b}{\frac{p}{a}-\frac{q}{b}}\right)$.

Replacing b with $-b$, we find the coordinates of the point of intersection of ① and another asymptote $y=-\frac{b}{a}x$.

$$\left(\frac{a}{\frac{p}{a}+\frac{q}{b}},\ -\frac{b}{\frac{p}{a}+\frac{q}{b}}\right)$$

Hence the area of $\triangle$OST is

$$\frac{1}{2}\left|\frac{a}{\frac{p}{a}-\frac{q}{b}}\times\frac{(-b)}{\frac{p}{a}+\frac{q}{b}}-\frac{a}{\frac{p}{a}+\frac{q}{b}}\times\frac{b}{\frac{p}{a}-\frac{q}{b}}\right|$$

$$=\frac{1}{2}\left|\frac{-2ab}{\frac{p^2}{a^2}-\frac{q^2}{b^2}}\right| = \frac{ab}{\left|\frac{p^2}{a^2}-\frac{q^2}{b^2}\right|}$$

Since the point $(p,\ q)$ lies on the hyperbola, the value of the denominator is 1. So the area is ab, independent of point A.

Therefore, the area of $\triangle$OST is a constant, independent of point A.

4

(1) From the condition, $A^4-E=O \quad \cdots ①$

Using the Cayley-Hamilton theorem, we have

$A^2=pA-qE$ (where $p=a+d$ and $q=ad-bc$) $\cdots ②$

Substitute ② into ①, we have

$$O=A^4-E$$
$$=(pA-qE)^2-E$$
$$=p^2A^2-2pqA+(q^2-1)E$$

Substitute ② again,

$$O=p^2(pA-qE)-2pqA+(q^2-1)E$$
$$=p(p^2-2q)A-(p^2q-q^2+1)E$$

and

$$p(p^2-2q)A=(p^2q-q^2+1)E \quad \cdots ③$$

If $p(p^2-2q)\neq 0$, then A is a constant multiple of identity matrix E : $A=kE$, where k is a real number. Then $k^4=1$ from ①, hence $k=\pm 1$.

However it cannot be accepted because $A\neq\pm E$.

Hence $p(p^2-2q)=0$, so $p=0$ or $p^2=2q$.

When $p^2=2q$,

(The coefficient of E in ③)$=q^2+1>0$,

and it does not satisfy ③. Therefore

$p=a+d=0 \cdots ④$

(Answer) $a+d=0$

(2) Substitute ④ into ③. We have

$$O=(1-q^2)E \quad \therefore\ q=ad-bc=\pm 1$$

From ④, we get $d=-a$. When $bc\geq 0$

$$ad-bc=-a^2-bc\leq 0$$

Hence $ad-bc=-1$ and

$$-a^2-bc=-1 \quad \cdots ⑤$$

So we have $bc=1-a^2\leq 1$

Since $a=\pm\sqrt{1-bc}$ from ⑤,

when $0\leq bc\leq 1$, the range of a is

$$-1\leq a\leq 1$$

(Answer) $-1\leq a\leq 1$

5

$x=y=z=0$ satisfies the equality

$$x^2+y^2+z^2=2xyz \quad \cdots①$$

If at least one of x, y and z is 0, the right side of ① is 0. So $x=y=z=0$.

In the other case, the left side of ① is positive. If there is a solution, there are two possibility that two of x, y and z are negative and the other is positive, or three of them are positive. Since the absolute values of the unknown are also the solution, x, y and z can be regarded as positive.

Since x, y and z are integers, the right side of ① is even. Hence x, y and z are all even, or two of them are odd and the other is even.

In the latter case, the remainder when the left side of ① is divided by 4 is 2. However the right side is divisible by 4. So it is not satisfied. Hence x, y and z are all even. Let $x=2x'$, $y=2y'$ and $z=2z'$, then we have

$$(x')^2+(y')^2+(z')^2=4x'y'z'$$

By the same argument, x', y' and z' must be even. If we continue this, it becomes that positive integers can be limitlessly divisible by 2, which leads a contradiction. Therefore there is no solution other than $x=y=z=0$.

Therefore the solution for ① is only $x=y=z=0$.

(Answer) $(x,\ y,\ z)=(0,\ 0,\ 0)$

6

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=1 \quad \cdots①$$

Multiply both sides by abc, we get

$$(a+b+c)(bc+ca+ab)=abc.$$

Expand the left side,

$$abc+ca^2+a^2b+b^2c+abc+ab^2$$
$$+bc^2+c^2a+abc=abc$$

Then,

$$(b+c)a^2+(b^2+c^2+2bc)a+b^2c+bc^2=0$$
$$(b+c)a^2+(b+c)^2a+bc(b+c)=0$$
$$(b+c)\{a^2+(b+c)a+bc\}=0$$
$$(b+c)(a+b)(a+c)=0$$

Hence $a+b=0$ or $b+c=0$ or $c+a=0$.

7

(1) $\displaystyle\int_1^t \sqrt{x^2-1}\,dx$

$$=at\sqrt{t^2-1}+b\log_e(t+\sqrt{t^2-1})$$

Differentiate both sides with respect to t,

$$\sqrt{t^2-1}$$
$$=a\sqrt{t^2-1}+at\cdot\frac{t}{\sqrt{t^2-1}}+b\cdot\frac{1+\frac{t}{\sqrt{t^2-1}}}{t+\sqrt{t^2-1}}$$
$$=\frac{a(t^2-1)+at^2+b}{\sqrt{t^2-1}}=\frac{2at^2-a+b}{\sqrt{t^2-1}}$$

Simplify,

$$t^2-1=2at^2-a+b$$
$$(1-2a)t^2+a-b-1=0$$

Since this is the identity for $t\,(\geq 1)$, we see

$$1-2a=0 \text{ and } a-b-1=0.$$

Hence $a=\dfrac{1}{2}$ and $b=a-1=-\dfrac{1}{2}$.

(Answer) $a=\dfrac{1}{2}$, $b=-\dfrac{1}{2}$

(2) Draw the perpendicular line BH from point B(p, $\sqrt{p^2-1}$) to the x-axis. The area bounded by OA, OB and the curve is

$$(\text{area of }\triangle\text{OBH})-\int_1^p\sqrt{x^2-1}\,dx$$
$$=\frac{1}{2}p\sqrt{p^2-1}-\int_1^p\sqrt{x^2-1}\,dx \quad \cdots①$$

From the result of (1),

$$\int_1^p\sqrt{x^2-1}\,dx=\frac{1}{2}p\sqrt{p^2-1}-\frac{1}{2}\log_e(p+\sqrt{p^2-1})$$

So ① equals $\dfrac{1}{2}\log_e(p+\sqrt{p^2-1})$.

Let this expression be $\dfrac{S}{2}$, we have

$$\log_e(p+\sqrt{p^2-1})=S$$
$$p+\sqrt{p^2-1}=e^S \quad \therefore \quad \sqrt{p^2-1}=e^S-p$$

Squaring both sides, we have $2pe^S=e^{2S}+1$

Therefore $p=\dfrac{e^{2S}+1}{2e^S}=\dfrac{e^S+e^{-S}}{2}$.

(Answer) $p=\dfrac{e^S+e^{-S}}{2}$